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THE GRAETZ PROBLEM IN COCURRENT-FLOW, DOUBLE-PIPE, HEAT EXCHANGERS

by

Ralph P. Stein

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DOUBLE-PIPE, HEAT EXCHANGERS

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NOMENCLATURE

Dimensional Quantities

a_1	Radius of circular tube, ft
a_2	Width of annular space, ft
b	Thickness of circular tube wall, ft
b'	Effective thickness of circular tube wall, ft [see equation (5a)]
c_i	Heat capacity of fluid "i," Btu/(lbm)(°F)
D_i	Hydraulic equivalent diameter, ft
h_i	Heat transfer coefficient for channel "i," Btu/(hr)(ft ²)(°F)
k_i	Thermal conductivity of fluid "i," Btu/(hr)(ft)(°F)
k_w	Thermal conductivity of circular tube wall, Btu/(hr)(ft)(°F)
ℓ	Axial distance or heat exchanger length measured from inlet, ft
q_i	Heat flux density at heat transfer surface of channel "i," Btu/(hr)(ft ²)
r	Radial distance from center of circular tube, ft
t_i	Local temperature of fluid "i," °F
t_∞	Equilibrium temperature, °F
t_{Bi}	Bulk temperature of fluid "i," °F
t_{0i}	Inlet temperature for fluid "i," °F
Δt_0	Inlet temperature difference, $t_{20} - t_{10}$, °F
u_i	Local velocity of fluid "i," ft/hr
\bar{u}_i	Average velocity of fluid "i," ft/hr
U_i	Overall heat transfer coefficient based on heat transfer area of channel "i," Btu/(hr)(ft ²)(°F)
W_i	Mass flow rate of fluid "i," lbm/hr
y	Distance normal to heat transfer surface of annular space, measured from insulated wall, ft
α_i	Thermal diffusivity of fluid "i," ft ² /hr

Dimensionless Quantities

g_i	u_i/\bar{u}_i
H	Heat-capacity flow-rate ratio, $c_2 W_2/(c_1 W_1)$
K	Relative thermal resistance of fluid; see equation (22)
K_w	Relative thermal resistance of wall; see equation (23)
Pe_i	Peclet number for fluid "i"
Nu_i	Nusselt number for channel "i"
Nu_i^0	Overall Nusselt number based on channel "i"
\bar{Nu}_i^0	Average or effective overall Nusselt number for use in heat exchanger design equation
x	Distance normal to heat transfer surface; see equations (11)
z	Axial distance measured from exchanger inlet; see equation (14)
ϵ	Heat exchanger effectiveness
ξ_i	Local temperature of fluid "i," $(t_i - t_{01})/\Delta t_0$
ξ_{Bi}	Bulk temperature of fluid "i," $(t_{Bi} - t_{01})/\Delta t_0$
ϕ	Effectiveness coefficient; see equations (54)
ω^2	$(KH)/2$

Subscripts

- $i = 1$, identifies circular tube
 $i = 2$, identifies annular space

THE GRAETZ PROBLEM IN COCURRENT-FLOW, DOUBLE-PIPE, HEAT EXCHANGERS

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ABSTRACT

It is shown that the Graetz Problem pertaining to the cocurrent-flow, double-pipe, heat exchanger can be studied analytically by extensions of familiar mathematical techniques. These techniques are used to derive a formal analytical solution of the problem, and numerical results are obtained. The results demonstrate the following: (1) Fully developed, laminar-flow, heat transfer ("film") coefficients in cocurrent-flow, double-pipe, heat exchangers depend upon the operating conditions of the exchanger, and their values can be significantly less than the value corresponding to the boundary condition of uniform wall temperature. (2) Use of customary design equation applied to laminar-flow heat exchangers can result in large errors even when the heat transfer is fully developed, and these errors will be largest when actual, fully developed, heat transfer coefficients are used.

I. INTRODUCTION

Analytical investigations of steady-state heat exchange between a fluid in fully developed laminar flow and the walls of the duct through which it flows constitute one of the most frequently occurring types of study published in the heat-transfer literature. Since Graetz, in 1885, appears to have been the first to publish such an investigation, the general category of these studies is referred to as "Graetz Problems." The classical Graetz Problem considers fully developed laminar flow through a uniform-temperature duct of circular cross section. Heat conduction within the fluid in the direction of flow, and heat generation resulting from viscous dissipation, are assumed to be negligible.

Many extensions of the classical Graetz Problem have appeared in the literature since Graetz's paper.⁽²⁾ The extensions include (1) replacing the uniform duct-temperature boundary condition with a uniform heat flux at the duct walls; (2) removing the assumptions of negligible axial-fluid heat conduction and

viscous dissipation; (3) replacing flow through a circular duct with flow through a duct with a cross section approximated by the space between infinitely wide, parallel planes; (4) illustrations of the use of Duhamel's theorem and superposition to generate solutions for nonuniform temperature and heat-flux boundary conditions in various combinations. One of the most recent of these extensions is that of Lundberg *et al.*⁽³⁾ who studied flow through annular spaces of various diameter ratios with all possible combinations of specified temperature and heat-flux boundary conditions.

The "fully developed, turbulent-flow" Graetz Problem has received considerable attention, as well as the "nonfully developed flow" case. Ducts with "two-dimensional" cross sections have also been studied.

With one exception, all of these extensions consider boundary conditions of specified duct wall temperature or heat flux. The exception is the somewhat more general condition of specifying that the wall temperature is proportional to the heat flux at the wall, and was used by Schenk and Beckers⁽⁸⁾ to approximate the case of heat transfer from the fluid through the duct wall to a uniform temperature environment. From a practical point of view, the specified wall-temperature boundary condition is the least important, since its use as a representation of an actual condition in a duct is applicable only for certain special limiting cases. Boundary conditions that specify heat fluxes apply more directly to many actual situations - e.g., in coolant passages of nuclear reactors. But what about the conditions that would apply to double-pipe heat exchangers? Except perhaps for certain special circumstances, none of the three types of boundary conditions mentioned above would be truly applicable.

In view of the practical importance of heat exchangers in general, it is perhaps surprising that investigations of applicable Graetz type problems are not contained in the literature. As will be seen, the formulation of such a problem for a cocurrent-flow, double-pipe, heat exchanger is relatively simple. Why then has this type of extension of the Graetz Problem not been investigated? A possible answer to this question appears to be related to the applicability of the classical mathematical techniques used to obtain analytical solutions to Graetz and similar problems. It is by no means immediately apparent that these same techniques can be applied to the more complicated "two-region" problem that results for the double-pipe heat exchanger.

The principal purpose of this paper is to show that these familiar mathematical techniques are applicable to the cocurrent-flow, double-pipe, heat-exchanger Graetz Problem, although in somewhat unfamiliar form. Use of these techniques will then be illustrated by computations of fully developed heat-transfer coefficients, and their dependence on the operating conditions of the exchanger will be briefly studied. In addition, a new heat-exchanger design quantity, called the "effectiveness coefficient," will be introduced, and its significance illustrated.

II. SOLUTIONS OF GRAETZ PROBLEMS

Graetz problems are formulated by expressing the heat-convection equation in a convenient coordinate system with appropriate simplifications, boundary conditions, and an "initial" condition describing the fluid temperature distribution at the inlet to the duct. With a few recent exceptions, and when interest is directed at conditions that are not too close to the duct inlet, formal solutions are obtained by the classical method of separation of variables. With the three types of boundary conditions mentioned in Section I, a familiar Sturm-Liouville system results, the solution of which provides an infinite sequence of orthogonal functions (eigenfunctions) with corresponding eigenvalues. Expansion of the initial condition as an infinite series of the eigenfunctions gives the coefficients necessary to complete the solution.

Except for certain cases with the space between infinite parallel planes,⁽⁷⁾ a general solution of the Sturm-Liouville equation of the system is not available in terms of known tabulated functions. As a result, approximation techniques are necessary in order to obtain numerical results from the formal solutions, and several of these are reported in the literature.

As will be seen, separation of variables applied to the cocurrent-flow, double-pipe, heat-exchanger problem results in the equivalent of a Sturm-Liouville system consisting of two Sturm-Liouville equations coupled at a common boundary. The coupling conditions yield boundary conditions that are not familiar for a Sturm-Liouville system. It will be shown, however, that solutions of the two-region problem form an infinite sequence of functions with corresponding eigenvalues, and with the equivalent of an orthogonality condition defined over both regions. The two-region orthogonality condition makes it possible to expand the initial condition for each region as an infinite series of appropriate eigenfunctions, thereby determining the coefficients necessary to complete the formal solution by separation of variables.

Recently, Nigram and Agrawal⁽⁶⁾ published accurate approximate solutions of the classical Graetz Problem for the space between infinitely wide parallel planes which were obtained by use of Biot's⁽¹⁾ variational formulation of the heat-conduction equation. With this approach, approximations are made directly to solutions of the heat-convection equation rather than to the Sturm-Liouville system resulting from the separation of variables. The accuracy of the approximation, however, is not always easy to judge. The technique is also applicable to the heat-exchanger Graetz Problem, and is presently being studied for this purpose.

III. FORMULATION OF HEAT EXCHANGER PROBLEM

The usual double-pipe heat exchanger consists of two concentric circular pipes with fluids flowing through the annular space and the central tube. In a cocurrent-flow exchanger, the fluids enter their respective flow channels at the exchanger inlet with different temperatures, transferring heat through the common wall as they flow in parallel along the length of the exchanger. A schematic diagram of this simple type of heat exchanger is illustrated in Figure 1, which also identifies some of the nomenclature to be used.

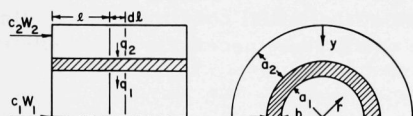


Fig. 1

Double-pipe Heat Exchanger,
Schematic Diagram

For the purpose of analysis, the following idealizations are made:

- (1) At the inlet to the duct, the temperature distributions within the fluids are uniform.
- (2) Physical properties are temperature-independent.
- (3) Frictional heating (viscous dissipation) is negligible.
- (4) Longitudinal heat conduction in the heat exchanger walls is negligible.
- (5) Longitudinal heat conduction in the fluids is negligible.
- (6) The velocity distributions within the fluids correspond to fully developed laminar flow.
- (7) The annular space of the exchanger is narrow in the sense that it can be approximated by the space between infinitely wide parallel planes.

The above idealizations are familiar ones and need little discussion. Removing all but the second and sixth results in only slight to moderate additional complications.

Let quantities associated with flow in the circular tube be identified by subscript "1," and those associated with flow in the narrow annular space by identified by subscript "2." The appropriate steady-state convection equations are then

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial t_1}{\partial r} \right) = \frac{u_1(r)}{\alpha_1} \frac{\partial t_1}{\partial \ell}, \quad (1a)$$

$$t_1(r, \ell), \quad 0 \leq r \leq a_1,$$

for the circular tube, and

$$\frac{\partial^2 t_2}{\partial y^2} = \frac{u_2(y)}{\alpha_2} \frac{\partial t_2}{\partial \ell}, \quad (1b)$$

$$t_2(y, \ell), \quad 0 \leq y \leq a_2,$$

for the narrow annular space. In the above, α represents the thermal diffusivity, and u the local fluid velocity in the axial or ℓ direction. For fully developed laminar flow,

$$u_1 = 2\bar{u}_1 [1 - (r/a_1)^2], \quad (2a)$$

and

$$u_2 = 6\bar{u}_2(y/a_2)[1 - (y/a_2)], \quad (2b)$$

where the overscore indicates the usual volumetric-flow average velocity.

If the directions of r and y are as indicated on Figure 1, application of the heat-flow boundary conditions results in

$$\left. \frac{\partial t_1}{\partial r} \right|_{r=0} = 0, \quad (3a)$$

$$\left. \frac{\partial t_2}{\partial y} \right|_{y=0} = 0, \quad (3b)$$

and

$$a_1 k, \left. \frac{\partial t_1}{\partial r} \right|_{r=a_1} = -(a_1 + b) k_2 \left. \frac{\partial t_2}{\partial y} \right|_{y=a_2}, \quad (4)$$

where k represents the thermal conductivity, and b the thickness of the circular tube wall. A fourth boundary condition is obtained by accounting for heat conduction through the wall. For this purpose, the familiar log-mean conduction equation is written as

$$-k_1 \left. \frac{\partial t_1}{\partial r} \right|_{r=a_1} = \frac{k_w}{b'} [t_1(a_1, \ell) - t_2(a_2, \ell)], \quad (5)$$

with

$$\begin{aligned} b' &= a_1 \ln(1 + b/a_1) \\ &= b[1 - (1/2)(b/a_1) + (1/3)(b/a_1)^2 + \dots]. \end{aligned} \quad (5a)$$

Finally, the "initial" condition is written as

$$t_1(r, 0) = t_{01} \quad (6a)$$

and

$$t_2(r, 0) = t_{02} \quad (6b)$$

Equations (1) through (6) represent a formal statement of the cocurrent-flow, double-pipe, heat-exchanger Graetz Problem. An additional relation, implicit in the above formulation and expressing a simple heat balance, is also of interest. To write this relation, the heat-content mean or bulk temperatures t_{B1} and t_{B2} are defined in the usual manner by

$$t_{B1} = \frac{2}{a_1^2 \bar{u}_1} \int_0^{a_1} u_1 t_1 r dr, \quad (7a)$$

and

$$t_{B2} = \frac{1}{a_2 \bar{u}_2} \int_0^{a_2} u_2 t_2 dy. \quad (7b)$$

The heat-balance relation is then written as

$$c_1 W_1 (t_{B1} - t_{01}) = c_2 W_2 (t_{02} - t_{B2}). \quad (8)$$

Dimensionless Formulation

It is convenient at this point to introduce a dimensionless formulation, and then restate the problem in purely mathematical form. This will serve not only to simplify the nomenclature, but also, as the equivalent of a dimensional analysis, to assist in identifying the important parameters of the problem.

First, note that as the heat-exchanger length increases without limit, both fluid temperatures must approach an equal and uniform equilibrium value, t_∞ . From equation (8),

$$t_\infty = t_{01} + [H/(1+H)] \Delta t_0, \quad (9)$$

where

$$H = c_2 W_2 / (c_1 W_1),$$

and

$$\Delta t_0 = t_{02} - t_{01}.$$

The heat-capacity flow-rate ratio, H , is a familiar one in heat-exchanger analyses and will be retained as a dimensionless parameter. Equation (9) suggests the dimensionless temperature

$$\xi_i = (t_i - t_{01})/\Delta t_0, \quad i = 1, 2. \quad (10)$$

The following dimensionless space variables are now introduced: dimensionless distances x , normal to the heat-transfer surfaces, defined by

$$x = r/a_1 \quad (11a)$$

for the circular tube, and

$$x = y/a_2 \quad (11b)$$

for the annular space; and a dimensionless length z , referenced arbitrarily to the properties of channel "1," defined by

$$z = \alpha_1 \ell / (\bar{u}_1 a_1^2). \quad (12)$$

Note that $0 \leq x \leq 1$, with $x = 1$ identifying the heat-transfer surface for all channels, and therefore subscripts are not required. Note also that $\xi_i(x, z)$ now replaces $t_1(r, \ell)$ and $t_2(y, \ell)$ and that $\xi_1(x, 0) = 0$, while $\xi_2(x, 0) = 1$. Further, from equation (9),

$$\lim_{z \rightarrow \infty} \xi_i(x, z) = \frac{H}{1 + H}. \quad (13)$$

The dimensionless length z can be expressed in more familiar terms by introducing the Peclet number defined by

$$Pe_i = \frac{D_i \bar{u}_i}{\alpha_i},$$

where D_i is the hydraulic equivalent diameter, equal to $2a_1$ for both channels. Introduction of the Peclet number into equation (12) results in an alternate definition for z given by

$$z = \frac{4}{Pe_1} \left(\frac{\ell}{D_1} \right). \quad (14)$$

The inverse of z is proportional to the Graetz number.

Substitution of the dimensionless variables ξ_i , x , and z for their dimensional equivalents in the differential equations and boundary conditions given by equations (1) to (6) results in the following dimensionless mathematical statement of the problem.

Differential Equations

$$\frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial \xi_1}{\partial x} \right) = g_1(x) \frac{\partial \xi_1}{\partial z}; \quad (15a)$$

$$\frac{\partial^2 \xi_2}{\partial x^2} = \omega^2 g_2(x) \frac{\partial \xi_2}{\partial z}. \quad (15b)$$

Boundary Conditions

$$\left. \frac{\partial \xi_i}{\partial x} \right|_0 = 0, \quad i = 1, 2; \quad (16)$$

$$K \left. \frac{\partial \xi_1}{\partial x} \right|_1 + \left. \frac{\partial \xi_2}{\partial x} \right|_1 = 0; \quad (17)$$

$$K_w \left. \frac{\partial \xi_1}{\partial x} \right|_1 + \xi_1(1, z) - \xi_2(1, z) = 0. \quad (18)$$

Initial Conditions

$$\xi_1(x, 0) = 0, \quad (19a)$$

$$\xi_2(x, 0) = 1, \quad (19b)$$

where

$$g_1(x) = 2(1 - x^2), \quad (20a)$$

$$g_2(x) = 6x(1 - x), \quad (20b)$$

$$\omega^2 = (1/2)KH, \quad (21)$$

$$K = (k_1/a_1)(a_2/k_2)(1 + b/a_1)^{-1}, \quad (22)$$

$$K_w = (k_1/a_1)(b'/k_w). \quad (23)$$

The bulk temperatures in dimensionless form become

$$\xi_{B1}(z) = 2 \int_0^1 g_1 \xi_1 x dx, \quad (24a)$$

$$\xi_{B2}(z) = \int_0^1 g_2 \xi_2 dx, \quad (24b)$$

with the heat-balance relation

$$\xi_{B1} = H(1 - \xi_{B2}). \quad (25)$$

The simple heat balance also implies the relations

$$\left. \frac{\partial \xi_1}{\partial x} \right|_1 = \frac{1}{2} \frac{d \xi_{B1}}{dz}, \quad (26a)$$

and

$$\left. \frac{\partial \xi_2}{\partial x} \right|_1 = \omega^2 \frac{d \xi_{B2}}{dz}, \quad (26b)$$

which can be obtained directly from equations (15) by integration with respect to x .

The parameter H is a familiar one in heat-exchanger analysis, as mentioned previously. The parameters K and K_w are not familiar ones but may be given simple physical significance when interpreted as relative thermal resistances. Thus, if the ratio a_1/k_1 is considered as a measure of the thermal resistance for heat flow to or from channel "1," K can be interpreted as the thermal resistance for heat flow to or from channel "2" relative to channel "1." Similarly, K_w can be interpreted as the relative thermal resistance of the exchanger common wall.

Equations (15) to (19) indicate that the dimensionless temperature distributions $\xi_i(x, z)$ depend exclusively on the three dimensionless groups H , K , and K_w in the sense that once these groups are assigned values, the mathematical solution is determined. Thus it is to be expected that quantities such as heat-exchange rates and heat-transfer coefficients, when expressed in appropriate dimensionless forms, will also depend on H , K , and K_w .

Heat Transfer Coefficients

The local, tube-side, heat-transfer coefficient is defined in the usual manner by

$$h_1(\ell) = \frac{k_1 \left. \frac{\partial t_1}{\partial r} \right|_{r=a_1}}{t_1(a_1, \ell) - t_{B1}(\ell)}, \quad (27a)$$

which, in dimensionless form, becomes

$$Nu_1(z) = \frac{2 \left. \frac{\partial \xi_1}{\partial x} \right|_1}{\xi_1(1, z) - \xi_{B1}(z)}, \quad (27b)$$

where Nu_1 is the Nusselt number defined by

$$Nu_1 = D_1 h_1 / k_1.$$

The same relations also apply to the local, annular-space, Nusselt number when subscripts "1" are replaced by subscripts "2."

Use of equations (26) gives the following somewhat more convenient relations:

$$Nu_1(z) = \frac{\frac{d\xi_{B1}}{dz}}{\xi_1(1, z) - \xi_{B1}(z)}, \quad (28a)$$

$$Nu_2(z) = \frac{2\omega^2 \frac{d\xi_{B2}}{dz}}{\xi_2(1, z) - \xi_{B2}(z)}. \quad (28b)$$

The overall heat-transfer coefficient, based on the heat-transfer area of the tube side of the exchanger, is defined by

$$U_1(\ell) = \frac{k_1 \left. \frac{\partial t_1}{\partial r} \right|_{r=a_1}}{t_{B2}(\ell) - t_{B1}(\ell)}, \quad (29a)$$

which, in dimensionless form becomes

$$Nu_1^0(z) = \frac{2 \left. \frac{\partial \xi_1}{\partial x} \right|_1}{\xi_{B2}(z) - \xi_{B1}(z)}, \quad (29b)$$

where

$$Nu_1^0 = \frac{D_1 U_1}{k_1}.$$

The quantity Nu_1^0 is an overall heat-exchanger Nusselt number based on tube-side properties. A more convenient expression for Nu_1^0 results upon use of the heat-balance relations, equations (25) and (26); viz.:

$$Nu_1^0(z) = \frac{\frac{d\xi_{B1}}{dz}}{1 - \frac{1+H}{H}\xi_{B1}(z)}. \quad (29c)$$

It is relatively simple to show that the familiar additive thermal-resistance concept for computing overall heat-transfer coefficients can be expressed in dimensionless form as

$$\frac{1}{Nu_1^0(z)} = \frac{1}{Nu_1(z)} + \frac{K_w}{2} + \frac{K}{Nu_2(z)}$$

and is consistent with the relations given for the various Nusselt numbers.

Heat Exchanger Effectiveness

The customary heat-exchanger effectiveness ϵ is defined as the ratio of the actual rate of heat transfer between fluids for a particular heat exchanger to the rate of heat transfer for a similar exchanger with infinite heat transfer area. For the cocurrent-flow, double-pipe exchanger, equations (10) and (13) lead to the simple relation

$$\epsilon = [(1+H)/H] \xi_{B1}, \quad (30)$$

so that equation (29c) can be written as

$$\frac{d}{dz} \ln(1 - \epsilon) = - \frac{1+H}{H} Nu_1^0(z). \quad (29d)$$

If Nu_1^0 is assumed to be independent of z , and equation (29d) is integrated from $z = 0$ to any value of z , a dimensionless equivalent of the customary heat-exchanger design relation results, viz.:

$$\epsilon \approx 1 - \exp\{-[(1+H)/H](\overline{Nu_1^0}z)\}, \quad (31)$$

where $\overline{Nu_1^0}$ represents the value of Nu_1^0 assumed constant during the integration.

IV. SOLUTION BY SEPARATION OF VARIABLES

Separation of variables applied to equations (15) through (18) leads to consideration of a solution in the form

$$\xi_1(x, z) = \frac{H}{1+H} + \sum_{n=1}^{\infty} C_n E_{1n}(x) e^{-\lambda_n^2 z} \quad (32a)$$

$$\xi_2(x, z) = \frac{H}{1+H} + \sum_{n=1}^{\infty} C_n E_{2n}(x) e^{-\lambda_n^2 z}, \quad (32b)$$

where the functions E_{1n} and E_{2n} must satisfy

$$\frac{1}{x} \frac{d}{dx} (x E'_{1n}) + g_1 \lambda_n^2 E_{1n} = 0, \quad (33a)$$

and

$$\frac{d^2 E_{2n}}{dx^2} + \omega^2 g_2 \lambda_n^2 E_{2n} = 0, \quad (33b)$$

with the boundary conditions

$$E'_{1n}(0) = 0, \quad (34a)$$

$$E'_{2n}(0) = 0, \quad (34b)$$

$$K E'_{1n}(1) + E'_{2n}(1) = 0, \quad (35)$$

$$K_w E'_{1n}(1) + E_{1n}(1) - E_{2n}(1) = 0. \quad (36)$$

In the above, primes denote differentiation with respect to x .

Since g_1 and g_2 are well-behaved functions of x , there is no doubt that solutions for E_{1n} and E_{2n} exist. To show that equations (32) represent a solution of the two-region boundary-value problem stated by equations (15) to (19), a relationship for the λ_n must be obtained, and it must be demonstrated that the coefficients can be determined so as to satisfy the initial conditions in the form

$$0 = \frac{H}{1+H} + \sum_{n=1}^{\infty} C_n E_{1n}(x), \quad (37a)$$

and

$$1 = \frac{H}{1+H} + \sum_{n=1}^{\infty} C_n E_{2n}(x). \quad (37b)$$

The homogeneous ordinary differential equations for E_{1n} and E_{2n} , together with the homogeneous boundary conditions given by equations (34), (35), and (36), constitute what might be called a "two-region" Sturm-Liouville problem. It will now be shown that this system of differential equations with boundary conditions has properties analogous to those of the more familiar Sturm-Liouville system, thus making it possible to determine the coefficients C_n so as to satisfy the initial conditions of the heat-exchanger Graetz Problem.

The Eigenvalue Equation

Let $F(\lambda, x)$ represent the solution of equation (33a) with $F_x(\lambda, 0) = 0$, and let $G(\lambda, x)$ represent the solution of equation (33b) with $G_x(\lambda, 0) = 0$, where the subscript "x" denotes partial differentiation with respect to x . Thus, with $A(\lambda)$ and $B(\lambda)$ arbitrary "constants" which might depend on λ , $AF(\lambda, x)$ and $BG(\lambda, x)$ also satisfy their respective differential equations. The boundary conditions given by equations (35) and (36) then require that $A(\lambda)$ and $B(\lambda)$ be chosen so that

$$KF_x(\lambda, 1)A + G_x(\lambda, 1)B = 0, \quad (38a)$$

and

$$[K_w F_x(\lambda, 1) + F(\lambda, 1)]A - G(\lambda, 1)B = 0. \quad (38b)$$

For this system of simultaneous homogeneous linear algebraic equations to have nonzero solutions for A and B , the coefficient determinant

$$\begin{vmatrix} KF_x(\lambda, 1) & G_x(\lambda, 1) \\ K_w F_x(\lambda, 1) + F(\lambda, 1) & -G(\lambda, 1) \end{vmatrix}$$

must be made equal to zero by proper choice of λ . This gives the eigenvalue equation

$$F(\lambda, 1)G_x(\lambda, 1) + KF_x(\lambda, 1)G(\lambda, 1) + K_w F_x(\lambda, 1)G_x(\lambda, 1) = 0. \quad (39)$$

From physical considerations, e.g., equation (13), and inspection of the assumed form of solution, equations (32), only real nonzero values of λ are required. Further, it should be possible to arrange the required roots, λ_n , of equation (39) in order of increasing size as

$$\lambda_1^2, \lambda_2^2, \lambda_3^2, \dots, \lambda_n^2, \dots,$$

and they should be (denumerably) infinite in number. For the more familiar Sturm-Liouville Problem (i.e., in a simple, finite, single region), these properties of the λ_n have been established rigorously with restrictions rarely of importance to physical problems. It should also be possible to establish these properties for the two-region problems, but this has not been attempted. Instead, it is noted that if the roots, λ_n , of equation (39) do not have the desired properties, then solutions of the form given by equation (32) are not proper.

The Eigenfunctions

Equation (38a) results in

$$B(\lambda) = -K \frac{F_x(\lambda, 1)}{G_x(\lambda, 1)} A(\lambda).$$

Since the system is homogeneous, either $A(\lambda)$ or $B(\lambda)$ can be chosen arbitrarily. The choice,

$$A(\lambda) = G_x(\lambda, 1),$$

results in convenient symmetry, for then the eigenfunctions E_{1n} and E_{2n} can be represented by

$$E_{1n}(x) = G_x(\lambda_n, 1)F(\lambda_n, x), \quad (40a)$$

and

$$E_{2n}(x) = -KF_x(\lambda_n, 1)G(\lambda_n, x), \quad (40b)$$

and clearly satisfy the two-region Sturm-Liouville system described by equations (33) to (36).

"Orthogonality" of the Eigenfunctions

The equivalent of an orthogonality condition for the E_{1n} and E_{2n} will now be established. The differential equations (33) are first manipulated for $n = i$ and $n = j$ in the manner used to derive properties of the familiar Sturm-Liouville system. Equation (33a) is written for $n = i$, and then for $n = j$, with $i \neq j$. The equation for $n = i$ is multiplied by E_{1j} , and the equation for $n = j$ is multiplied by E_{1i} . The two resulting equations are then subtracted, and the following is obtained:

$$E_{1j} \frac{d}{dx} (xE'_{1i}) - E_{1i} \frac{d}{dx} (xE'_{1j}) + (\lambda_i^2 - \lambda_j^2)g_1 E_{1i} E_{1j} x = 0.$$

Use of the identity

$$\frac{d}{dx} [x(E_{1j}E'_{1i} - E_{1i}E'_{1j})] \equiv E_{1j} \frac{d}{dx} (xE'_{1i}) - E_{1i} \frac{d}{dx} (xE'_{1j})$$

and multiplication by dx enable the above to be written as

$$d[x(E_{1j}E'_{1i} - E_{1i}E'_{1j})] + (\lambda_1^2 - \lambda_j^2) g_1 E_{1i} E_{1j} dx = 0.$$

Integration between $x = 0$ and $x = 1$ results in

$$(\lambda_j^2 - \lambda_1^2) \int_0^1 x g_1 E_{1i} E_{1j} dx = E_{1j}(1)E'_{1i}(1) - E_{1i}(1)E'_{1j}(1). \quad (41a)$$

In a similar manner, equation (33b) gives

$$\omega^2 (\lambda_j^2 - \lambda_1^2) \int_0^1 g_2 E_{2i} E_{2j} dx = E_{2j}(1)E'_{2i}(1) - E_{2i}(1)E'_{2j}(1). \quad (41b)$$

The integration to obtain equation (41b) uses the boundary condition $E'_{2n}(0) = 0$.

The integrals of equations (41) can be related to each other by using the coupling boundary conditions at $x = 1$. Thus, equation (35) produces

$$E'_{2n}(1) = -KE'_{1n}(1),$$

and equation (36) leads to

$$E_{2n}(1) = E_{1n}(1) + K_w E'_{1n}(1).$$

Use of these conditions results in the relation

$$E_{2j}(1)E'_{2i}(1) - E_{2i}(1)E'_{2j}(1) = -K[E_{1j}(1)E'_{1i}(1) - E_{1i}(1)E'_{1j}(1)]$$

which, with equations (41), implies that

$$\omega^2 \int_0^1 g_2 E_{2i} E_{2j} dx + K \int_0^1 x g_1 E_{1i} E_{1j} dx = 0.$$

With $\omega^2 = \frac{1}{2} KH$, the above can be written as

$$\int_0^1 (2xg_1E_{1i}E_{1j} + Hg_2E_{2i}E_{2j})dx = 0, \quad i \neq j. \quad (42)$$

Equation (42) is the equivalent of an orthogonality condition for the eigenfunctions E_{1n} and E_{2n} . Note, however, that the familiar correspondence of an orthogonal set of functions to the concept of an orthogonal set of vectors in a space of denumerably infinite dimensions is no longer apparent from equation (42). The correspondence can be retained by replacing x by $1 - x$ in region 1, and by $x - 1$ in region 2, and rephrasing the problem accordingly in the single region defined by $-1 \leq x \leq 1$. This type of formulation might be more convenient for theoretical investigations but offers no immediate advantage for the treatment given here.

For $i = j = n$, equations (33) lead to a normalizing factor defined by

$$N_n = \int_0^1 (2xg_1E_{1n}^2 + Hg_2E_{2n}^2)dx. \quad (43)$$

Note that $E_{1n} = E_{2n} = 1$ is a solution of the two-region Sturm-Liouville problem corresponding to $\lambda_n = 0$, and that

$$\int_0^1 (2xg_1E_{1i} + Hg_2E_{2i})dx = 0, \quad i \neq 0.$$

Thus to complete the set of functions E_{1n} and E_{2n} , the case of $n = 0$ with $\lambda_0 = 0$, and $E_{10} = E_{20} = 1$ must be included. Also, application of equation (43) gives

$$N_0 = 1 + H.$$

Expansions of Arbitrary Functions

Let $f_1(x)$ and $f_2(x)$ be arbitrary functions defined in regions 1 and 2, respectively, with $0 \leq x \leq 1$. To avoid concern about the validity of the operations to be performed with these functions, assume that they are continuous in their respective regions. Consider the expansions

$$f_1(x) = \sum_{n=0}^{\infty} C_n E_{1n}(x); \quad (44a)$$

$$f_2(x) = \sum_{n=0}^{\infty} C_n E_{2n}(x). \quad (44b)$$

Multiply the first expansion by $2xg_1E_{1j}dx$, and the second by $Hg_2E_{2j}dx$. Add the resulting expressions, and integrate between $x = 0$ and $x = 1$, using equations (42) and (43). The following formula for the C_n results:

$$C_n = \frac{1}{N_n} \int_0^1 [2xg_1E_{1n}f_1(x) + Hg_2E_{2n}f_2(x)]dx. \quad (45)$$

For $n = 0$, this becomes

$$C_0 = \frac{1}{1+H} \int_0^1 [2xg_1f_1(x) + Hg_2f_2(x)]dx. \quad (46)$$

Thus, the expansions of equations (44) are completely analogous to representations of arbitrary functions by a generalized Fourier series, at least formally.

Completion of Solution of Heat-exchanger Graetz Problem

The solution of the cocurrent-flow, heat-exchanger problem in the form of equation (32) can now be completed. To satisfy the initial conditions as required by equation (37), the expansions of equations (44) are written with $f_1(x) = 0$ and $f_2(x) = 1$. Thus, equation (45) results in

$$C_n = \frac{H}{N_n} \int_0^1 g_2E_{2n}dx. \quad (47)$$

Also, since

$$\int_0^1 g_2dx = 1,$$

equation (46) leads to

$$C_0 = H/(1+H), \quad (47a)$$

as required by equations (32).

It will be convenient for later application to define the quantities

$$B_{1n} = 2 \int_0^1 xg_1E_{1n}dx, \quad (48a)$$

and

$$B_{2n} = \int_0^1 g_2 E_{2n} dx. \quad (48b)$$

Application of equations (24) and (25), i.e., the heat-balance relation, to equations (32) results in

$$B_{1n} = -HB_{2n}, \quad n \neq 0.$$

(Note that the same relationship between the B_{1n} is implied by the orthogonality expression.) Thus, equation (47) can be written as

$$C_n = HB_{2n}/N_n; \quad (49a)$$

$$C_n = -B_{1n}/N_n, \quad n \neq 0. \quad (49b)$$

The integrations defining B_{1n} and B_{2n} can be completed with the assistance of equations (33) and the boundary conditions. The following equations result:

$$B_{1n} = -2E'_{1n}(1)/\lambda_n^2; \quad (50a)$$

and

$$B_{2n} = -E'_{2n}(1)/(\omega^2 \lambda_n^2). \quad (50b)$$

Fully Developed Heat-transfer Coefficients

For sufficiently large values of z , all but the $n = 1$ terms of equations (32) can be neglected. Application of this "large z " solution to equations (28) and (29c) give the following asymptotic or fully developed values of the various heat-transfer coefficients:

$$Nu_1(\infty) = \lambda_1^2 B_{11}/[B_{11} - E_{11}(1)]; \quad (51a)$$

$$Nu_1(\infty) = KH\lambda_1^2 B_{21}/[B_{21} - E_{21}(1)]; \quad (51b)$$

and

$$Nu_1^0(\infty) = [H/(1 + H)]\lambda_1^2. \quad (52)$$

Heat Exchanger Effectiveness

Application of equations (32) and (49) to the defining relation for the dimensionless bulk temperature ξ_{B1} , and introduction of equation (30), result in the expression

$$\epsilon(z) = 1 - \frac{1+H}{H} \sum_{n=1}^{\infty} \frac{B_{1n}^2}{N_n} e^{-\lambda_1^2 z} \quad (53)$$

for the heat-exchanger effectiveness. Multiplying and dividing by quantities related to the first eigenvalue, and using equation (52) result in the following form of equation (53):

$$\epsilon = 1 - \phi(z) e^{-\lambda_1^2 z} \quad (54a)$$

$$= 1 - \phi(z) \exp \left\{ - \left[(1+H)/H \right] \text{Nu}_1^0(\infty) z \right\}, \quad (54b)$$

where

$$\phi(z) = \frac{1+H}{H} \sum_{n=1}^{\infty} \frac{B_{1n}^2}{N_n} e^{-(\lambda_n^2 - \lambda_1^2)z}. \quad (55)$$

Comparison of this form with the customary heat-exchanger design relation as given by equation (31) shows that the two will give approximately equivalent results when $\overline{\text{Nu}}_1^0 \approx \text{Nu}_1^0(\infty)$ and $\phi(z) \approx 1$.

A useful relationship for the maximum variation of $\phi(z)$ can be determined. First note that for sufficiently large z ,

$$\begin{aligned} \lim_{z \rightarrow \infty} \phi(z) &= \phi(\infty) \\ &= [(1+H)/H](B_{11}^2/N_1), \end{aligned} \quad (56)$$

while the requirement that $\epsilon = 0$ when $z = 0$ gives

$$\phi(0) = 1.$$

Since B_{1n}^2 and N_n [see equation (43)] are positive numbers, and $\lambda_n < \lambda_{n+1}$, $\phi(z)$ must decrease monotonically with increasing z . Thus

$$\phi(\infty) \leq \phi(z) \leq 1.$$

This relationship, together with appropriate values of $\text{Nu}_1^0(\infty)$, can be used to test the accuracy of the customary heat-exchanger design equation [equation (31)].

The Effectiveness Coefficient

When $\phi(z)$ is significantly different from unity, it can become an important quantity for heat-exchanger design. For this reason it is called "The Effectiveness Coefficient." The quantity

$$\phi(\infty) = [(1 + H)/H](B_{11}^2/N_1) \quad (57)$$

is the "Fully Developed Effectiveness Coefficient" for the cocurrent-flow, double-pipe, heat exchanger.

V. THE PLUG-FLOW CASE

If the velocity distributions $u_1(r)$ and $u_2(y)$ are taken to be uniform, rather than parabolic, then $g_1 = g_2 = 1$, and the differential equations for the eigenfunctions, i.e., equations (33), have solutions in terms of familiar elementary functions. This simple velocity distribution represents the idealization of plug- or rod-like flow and has been used as a basis for an analysis of heat transfer between liquid metals in turbulent flow in cocurrent-flow, double-pipe, heat exchangers.⁽⁹⁾ Solutions for the appropriate, two-region, boundary-value problem - i.e., equations (15) to (19) with $g_1 = g_2 = 1$ - were obtained by Laplace transform techniques, and thus are available for comparison with solutions obtained by separation of variables.

Recall that $F(\lambda, x)$ represents the solution of equation (33a) with $F_x(\lambda, 0) = 0$ and that $G(\lambda, x)$ represents the solution of equation (33b) with $G_x(\lambda, 0) = 0$. For $g_1 = g_2 = 1$,

$$F(\lambda, x) = J_0(\lambda x); \quad (58a)$$

and

$$G(\lambda, x) = \cos(\omega \lambda x). \quad (58b)$$

Thus, if it is noted that

$$F_x(\lambda, x) = -\lambda J_1(\lambda x),$$

and

$$G_x(\lambda, x) = -\omega \lambda \sin(\omega \lambda x),$$

the eigenvalue relation equivalent to equation (39) becomes, after slight rearrangement,

$$J_0(\lambda) \sin(\omega \lambda) + (K/\omega) J_1(\lambda) \cos(\omega \lambda) - K_w \lambda J_1(\lambda) \sin(\omega \lambda) = 0. \quad (59)$$

The eigenvalues are given by the positive nonzero roots of equation (59). Application of equations (40) determines the eigenfunctions for the plug-flow case to be

$$E_{1n}(x) = -\omega \lambda_n \sin(\omega \lambda_n) J_0(\lambda_n x), \quad (60a)$$

and

$$E_{2n}(x) = K \lambda_n J_1(\lambda_n) \cos(\omega \lambda_n x). \quad (60b)$$

Substitution of equations (60) into equations (43) and (47) gives the required coefficients C_n and completes the plug-flow solution in the form of equations (32). It is relatively simple to show that the solution obtained by separation of variables is equivalent to the same form of solution obtained by Laplace transform techniques. For example, application of equations (50) and (51) results in, for the plug-flow case,

$$Nu_1(\infty) = \frac{2\lambda_1^2 J_1(\lambda_1)}{2J_1(\lambda_1) - \lambda_1 J_0(\lambda_1)},$$

and

$$Nu_2(\infty) = \frac{2(\omega\lambda_1)^2 \sin(\omega\lambda_1)}{\sin(\omega\lambda_1) - \omega\lambda_1 \cos(\omega\lambda_1)},$$

with λ_1 denoting the least, nonzero eigenvalue as determined from equation (59). These relations are the same as those given in reference (9).

VI. THE LAMINAR-FLOW CASE

For the laminar-flow case, i.e., with

$$g_1 = 2(1 - x^2),$$

and

$$g_2 = 6x(1 - x),$$

useful solutions for the differential equations (33) are not known in terms of tabulated functions. Thus, solutions by power-series representation of E_{1n} and E_{2n} , or by the use of approximation techniques, are necessary.

Several techniques are available for approximating eigenvalues and eigenfunctions, most (if not all) of which would be applicable to the two-region Sturm-Liouville Problem defined by equations (33) to (36). With the eigenfunction solution available for the plug-flow case, a convenient basis for an approximation method is to represent the laminar-flow eigenfunctions as a linear sum of plug-flow eigenfunctions. In this way the boundary conditions - i.e., equations (34) to (36) - are automatically satisfied. The coefficients multiplying each term in the sum are then determined so that the differential equations - i.e., equations (33) - are approximately satisfied. After some experimentation with other methods - especially the use of polynomial approximations for the eigenfunctions in a variational formulation of the problem (see Appendix B) - approximation by a linear sum of plug-flow eigenfunctions was chosen for detailed study.

This type of approximation method for eigenfunctions is a familiar one in perturbation analyses of the wave equations of theoretical physics.⁽⁵⁾ In such applications, the quantities $1 - g_1$ and $1 - g_2$ would represent small perturbations from a system for which the eigenfunctions and eigenvalues are known. The procedure was first applied to Graetz Problems by Millsaps and Pohlhausen⁽⁴⁾ with considerable success. Since the "perturbation" represented by the difference between a uniform velocity distribution (plug flow) and a parabolic velocity distribution is not really small, the procedure requires use of more terms in the sum than is usual in perturbation calculations. As a result, the technique when applied to Graetz-type Problems is not suitable for hand calculation, but does result in a procedure which is convenient for high-speed digital machine computation.

The procedure, as applied to the two-region Sturm-Liouville Problem described by equations (33) to (36), is developed in two different ways in Appendices A and B. In Appendix A, the orthogonality of the plug-flow solutions is used as the basis; in Appendix B, a variational formulation of the two-region Sturm-Liouville Problem is used.

The approximation by a linear sum of plug-flow eigenfunctions can be written as

$$E_{1n}(x) \approx \sum_{j=0}^r a_{nj} u_j(x), \quad (61a)$$

and

$$E_{2n}(x) \approx \sum_{j=0}^r a_{nj} v_j(x), \quad (61b)$$

where u_j represents the j th eigenfunction for region "1" for the plug-flow problem, and v_j represents the same for region "2"; i.e., u_j is given by equation (60a), and v_j by equation (60b). "Best" values of the coefficients a_{nj} , as well as of the laminar-flow eigenvalues λ_n , for $n = 1, 2, \dots, r$, are obtained as solutions of a corresponding eigenvalue problem related to a homogeneous set of $r + 1$ linear algebraic equations with a_{nj} as the unknowns. Details are presented in Appendix A. The corresponding eigenvalue problem is a relatively standard one for a high-speed digital computer, and solutions with r as large as 10 can be obtained quite rapidly.

Initially it was hoped that this approximation technique would make it possible to explore heat-exchanger operation over the entire likely range of the parameters H , K , and K_w . It was discovered, however, as discussed further in Appendix C, that convergence was not sufficiently rapid or uniform. In most instances the approximations exhibited oscillatory behavior with increasing r , making it difficult to judge the accuracy of results beyond the third significant figure, even for r as large as 16. Because of this, a compromise was necessary which limited the range of parameters covered, and allowed consideration of only those quantities related to the first and second eigenfunctions.

The results obtained, including an indication of their accuracy, are listed in Tables I, II, and III, and discussed with respect to general trends in Section VII.

In most cases, the accuracy appears to be of about the same order as to be expected from a precision electronic analog-computer solution of equations (33) to (36). Since use of an analog computer should permit a more rapid coverage of a much larger range of the parameters, and should allow removal of the narrow annular space idealization with only slight additional complications, this method of computation is being explored.

Table I

QUANTITIES RELATED TO FIRST EIGENFUNCTION

K_w	K	H	λ_1^2	$E_{11}(1)$	$-B_{11}$	N_1
0	1	0.1	10.94	0.3091	0.0389	0.0806
0	1	0.5	5.81	0.426	0.497	0.981
0	1	1	4.134	0.140	0.846	1.717
0	2	0.5	4.336	0.191	0.803	2.28
0	0.5	0.5	6.75	0.361	0.266	0.331
0	0.1	0.5	7.52	0.106	0.057	0.0184
0	0.01	0.5	7.688	0.01150	0.00579	0.00020
0	0.1	2	4.78	0.118	0.289	0.166
0	0.1	1	5.89	0.124	0.139	0.057
0	0.1	0.1	11.27	0.0353	0.00360	0.00099
0.1	0.1	0.5	7.02	0.0925	0.0608	0.0182
0.2	0.1	0.5	6.530	0.0784	0.06389	0.0181

Note: The results tabulated above are uncertain in the last digit given.

Table II

QUANTITIES RELATED TO SECOND EIGENFUNCTION

K_w	K	H	λ_2^2	$E_{12}(1)$	$-B_{12}$	N_2
0	1	0.1	30.4	0.39	0.097	0.24
0	1	0.5	18.32	-0.38	0.167	0.55
0	1	1	16.42	-0.24	0.056	0.11
0	2	0.5	15.16	-0.34	0.049	0.16
0	0.5	0.5	21.7	-0.059	0.22	0.60
0	0.1	0.5	25.7	0.12	0.078	0.074
0	0.01	0.5	36.64	0.02	0.008	0.0009
0	0.1	2	26.4	-0.0092	0.177	0.343
0	0.1	1	23.52	0.078	0.14	0.21
0	0.1	0.1	35.7	0.062	0.007	0.003
0.1	0.1	0.5	23.20	0.032	0.078	0.073
0.2	0.1	0.5	21.21	-0.036	0.071	0.065

Note: The results tabulated above are uncertain in the last digit given.

Table III

FULLY DEVELOPED NUSSELT NUMBERS AND
EFFECTIVENESS COEFFICIENTS

K_w	K	H	$Nu_1(\infty)$	$Nu_2(\infty)$	$Nu_1^0(\infty)$	$\phi(\infty)$
0	1	0.1	1.22	5.3	0.994	0.206
0	1	0.5	3.13	5.08	1.938	0.751
0	1	1	3.55	4.95	2.067	0.834
0	2	0.5	3.50	4.92	1.445	0.849
0	0.5	0.5	2.87	5.22	2.250	0.644
0	0.1	0.5	2.63	5.4	2.51	0.53
0	0.01	0.5	2.57	6	2.56	0.50
0	0.1	2	3.40	5.1	3.190	0.759
0	0.1	1	3.107	5.5	2.943	0.674
0	0.1	0.1	1.04	6	1.024	0.144
0.1	0.1	0.5	2.78	5.5	2.340	0.608
0.2	0.1	0.5	2.931	5.6	2.176	0.677

NOTE: The results tabulated above are uncertain in the last digit given.

VII. DISCUSSION OF NUMERICAL RESULTS

Since the principal intent of this report was to show that the cocurrent-flow, double-pipe, heat-exchanger, Graetz Problem can be studied by an extension of familiar mathematical techniques, the numerical results obtained will be discussed only briefly.

The range of the parameters H , K , and K_W explored include those values that are most likely for an actual laminar-flow, double-pipe heat exchanger, but probably require some future extension for completeness. Values of H usually will be near unity, values of K near 0.1, and values of K_W less than 0.1. In unusual situations, values of H from 0.1 to 10 are conceivable. With fluids on either side of the exchanger of significantly different thermal conductivities, values of K could also span a large range. With liquid metals, values of K_W could be significantly greater than 0.1, but rarely greater than unity.

Fully Developed Heat Transfer Coefficients

Figures 2, 3, and 4, prepared from the results in Table III, show graphs of tube-side, fully developed, heat-transfer coefficients as a function of exchanger operation. The heat-transfer coefficients have been normalized with respect to the value appropriate to the boundary condition of uniform heat flux at the tube wall. The normalized values were obtained by dividing the tube-side Nusselt number by the uniform flux value of 4.36.⁽²⁾ The normalized heat-transfer coefficient corresponding to the boundary condition of uniform surface temperature is located on the graphs for reference, and was obtained by dividing the appropriate Nusselt number, 3.65,⁽²⁾ by 4.36. Also shown on the graphs (by dashed lines) are the comparable results obtained previously for the plug-flow case.⁽⁹⁾

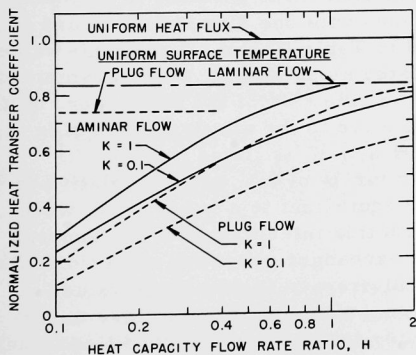


Fig. 2. Fully Developed, Tube-side Heat-transfer Coefficients, Normalized; Effect of Heat Capacity Flow Rate Ratio ($K_W = 0$)

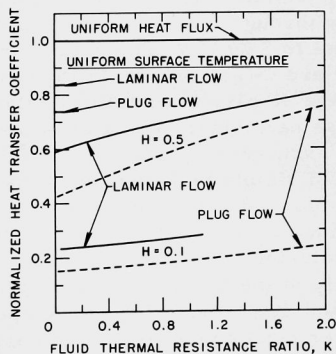


Fig. 3. Fully Developed, Tube-side Heat-transfer Coefficients, Normalized; Effect of Fluid Thermal Resistance ($K_W = 0$)

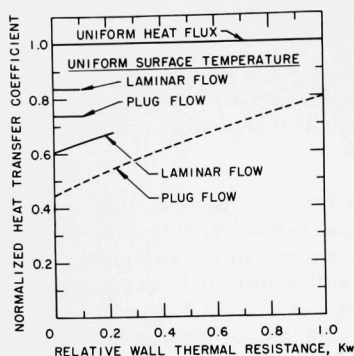


Fig. 4

Fully Developed, Tube-side Heat-transfer Coefficients, Normalized; Effect of Wall Thermal Resistance ($H = 0.5$, $K = 0.1$)

The results are similar to those found for the plug-flow case, and it can be surmised that the general conclusions concerning fully developed, plug-flow, heat-transfer coefficients apply to the laminar-flow case also. In particular, values of fully developed heat-transfer coefficients (1) can be significantly less than those corresponding to the boundary condition of uniform wall temperature; (2) depend upon the operating conditions of the exchanger as characterized by the parameters H , K , and K_w ; and (3) are influenced most by the heat-capacity flow-rate ratio H , especially when it is less than unity. For the plug-flow case, it was demonstrated that the heat-transfer coefficient can never be larger than the corresponding uniform-flux value, and that this value is approached as H , K , and K_w increase without limit. The graphs suggest that this general result applies to the laminar-flow case also, although further analysis and/or computation is required for verification.

Table III shows that over the range of the parameters explored, the fully developed, narrow, annular-space Nusselt numbers vary only slightly. The corresponding uniform-flux value pertains to laminar flow through infinitely wide, parallel planes with one side insulated and is equal to 5.39.⁽¹⁰⁾ The values presented in Table III are all near this value, and are larger only when they are indicated as uncertain. The parameters H and K have been defined relative to the tube side of the exchanger. But these parameters can also be defined relative to the annular space side of the exchanger, in which case values of H and K as listed in Table III would simply be inverted. Thus, a value for H of 0.1, defined relative to the tube side of the exchanger, would be equivalent to a value of 10, when defined relative to the annulus side. With this interpretation, the general conclusions concerning the effect of the exchanger operating conditions also apply to the results obtained for the annular space in that large values of H and K result in fully developed heat-transfer coefficients close to, but never larger than, the value corresponding to the boundary condition of uniform flux.

Fully Developed Effectiveness Coefficients

Figures 5 and 6 are graphs of the fully developed effectiveness coefficient $\phi(\infty)$ as a function of the parameters H , K , and K_W . Clearly, $\phi(\infty)$ is significantly less than unity within regions of practical interest, suggesting that it could be an important quantity for heat-exchanger design. To illustrate the significance of this quantity, a calculation was made to predict the required length of cocurrent, laminar-flow, double-pipe heat exchanger to obtain a desired heat-exchanger effectiveness of 0.6. The parameter H was taken as unity, K as 0.1, and K_W as zero. The results obtained by use of the customary design equation [equation (31)] with \overline{Nu}_1^0 computed from fully developed, uniform surface-temperature, heat-transfer coefficients, were compared with those obtained using equation (54). It was found that $\phi(z) = \phi(\infty)$ (i.e., the heat transfer is fully developed), and that the customary design equation predicted a length 52% longer than required. For $H = 0.5$, the error increased to 145%; for $H = 2$, the error decreased to 34%. For \overline{Nu}_1^0 based on fully developed, uniform heat-flux, heat-transfer coefficients, the errors were roughly halved; with \overline{Nu}_1^0 taken equal to $Nu_1^0(\infty)$ - i.e., based on the actual, fully developed, overall, heat-transfer coefficients - the errors increased to 230% for $H = 0.5$, 75% for $H = 1$, and 42% for $H = 2$. For an effectiveness of 0.8, however, the errors in the customary design equation were nearly negligible, except when using $\overline{Nu}_1^0 = Nu_1^0(\infty)$.

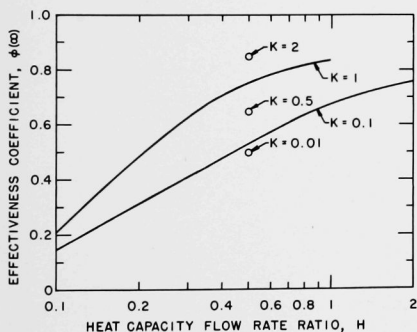


Fig. 5. Effect of Operating Conditions on Fully Developed Effectiveness Coefficient ($K_W = 0$)

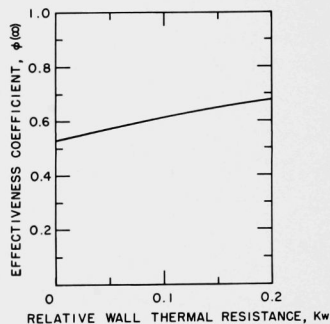


Fig. 6. Effect of Wall Thermal Resistance on Fully Developed Effectiveness Coefficient ($H = 0.5$, $K = 0.1$)

The effectiveness coefficient accounts for the high rates of heat transfer occurring in the thermal-entrance region of the exchanger. The customary design equation does not account for these high rates except, perhaps, by use of a suitably increased, average, overall heat-transfer coefficient. Thus, results using the design equation are best when \overline{Nu}_1^0 is

calculated from fully developed, uniform-flux, heat-transfer coefficients, since this basis for the calculation gives the largest value of \overline{Nu}_1^0 consistent with the design-equation assumption that it be independent of exchanger length.

The results suggest that for sufficiently small values of the heat-exchanger effectiveness (say less than 0.8), accurate design of laminar-flow heat exchangers can not be based on the customary design equation, even when the heat transfer is fully developed. The use of the effectiveness coefficient and equations (54) for cocurrent flow offers a possible alternate method of calculation, but is complicated by the dependence of $\phi(\infty)$ and $Nu_1^0(\infty)$ on the exchanger operating conditions. Future study appears warranted.

VIII. SUMMARY

It has been shown that the Graetz Problem pertaining to the cocurrent-flow, double-pipe, heat exchanger can be studied analytically by extensions of familiar mathematical techniques. When these techniques were applied to the corresponding plug-flow problem, the analytical solution obtained agrees with a solution obtained previously by Laplace transform methods. When the techniques were applied to the laminar-flow problem, a well-known approximation method was used to obtain numerical results of interest with moderate success.

Future work on the heat-exchanger Graetz Problem will explore use of a precision electronic-analog computer to extend the results reported here.

APPENDIX A

A Formal Algebraic Solution for the Laminar-flow Eigenfunctions

To simplify the nomenclature somewhat, the plug-flow eigenfunctions as given by equation (60) are represented by

$$u_n(x) = -\omega\gamma_n \sin(\omega\gamma_n)J_0(\gamma_n x)$$

for region 1, and

$$v_n(x) = K\gamma_n J_1(\gamma_n) \cos(\omega\gamma_n x)$$

for region 2, where γ_n represents the eigenvalues given by the nonzero roots of equation (59) with γ replacing λ , and $n = 1, 2, 3, \dots$. To complete the set of functions, the definition $u_0 = v_0 = 1$ must be included.

The eigenfunctions of the laminar-flow problem, E_{1n} and E_{2n} , are represented by the expansions of equations (44) written as

$$E_{1n}(x) = \sum_{j=0}^{\infty} a_{nj} u_j(x), \quad (\text{A-1a})$$

and

$$E_{2n}(x) = \sum_{j=0}^{\infty} a_{nj} v_j(x). \quad (\text{A-1b})$$

These relations satisfy the boundary conditions of the laminar-flow, "two-region," Sturm-Liouville Problem. When these expansions are substituted into differential equations (33), and use is made of the relations

$$\frac{1}{x} \frac{d}{dx} (xu_j') = -\gamma_j^2 u_j$$

and

$$\frac{d^2 v_j}{dx^2} = -\omega^2 \gamma_j^2 v_j,$$

the following equations result:

$$\sum_{j=0}^{\infty} a_{nj} [g_1(x) \lambda_n^2 - \gamma_j^2] u_j(x) = 0, \quad (\text{A-2a})$$

and

$$\sum_{j=0}^{\infty} a_{nj} [g_2(x) \lambda_n^2 - \gamma_j^2] v_j(x) = 0, \quad (\text{A-2b})$$

with $\gamma_0 = 0$ by definition. Equation (A-2a) is now multiplied by $2xu_k$, and equation (A-2b) by Hv_k . The results are added and integrated between $x = 0$ and $x = 1$, making use of the orthogonality condition of equation (42). The following equation results:

$$\lambda_n^2 \sum_{j=0}^{\infty} a_{nj} P_{kj} = \gamma_k^2 a_{nk} N_k^{(0)}, \quad (\text{A-3})$$

$$k = 0, 1, 2, 3, \dots,$$

where

$$\begin{aligned} P_{kj} &= \int_0^1 (2xg_1 u_k u_j + Hg_2 v_k v_j) dx \\ &= P_{jk}, \end{aligned} \quad (\text{A-4})$$

and

$$N_k^{(0)} = \int_0^1 (2xu_k^2 + Hv_k^2) dx. \quad (\text{A-5})$$

Note that $P_{\infty} = 1 + H$. For this system of homogeneous equations to have nonzero solutions for the a_{nk} , the λ_n must be chosen so that the coefficient determinant is zero. Since the determinant is of infinite order, setting it equal to zero will in principle determine an infinite set of values of λ_n corresponding to the eigenvalue equation for the laminar flow problem. For each λ_n , there is a corresponding solution of equations (A-3) which determines the a_{nk} , $k = 0, 1, 2, \dots$. Since the system is homogeneous, one of the a_{nk} can be chosen arbitrarily. Thus, in a formal way, equations (A-1), together with equation (A-3), represent an exact algebraic solution for the laminar-flow eigenfunctions, provided no difficulties are associated with the convergence of equations (A-1) and of the infinite determinant.

Successful use of the algebraic solution as a basis for numerical calculations depends on proper convergence, and as a result can also serve as a direct test of convergence. Thus, as an approximation, equations (A-1) are applied by using only the first $1 + r$ terms, i.e., $j = 0, 1, 2, \dots, r$, and the behavior of the results is studied as r is increased successively. The results of this type of study are described in Appendix C.

APPENDIX B

A Variational Approach

Further confidence in the approximation based on the use of truncated forms of equations (A-1) can be obtained analytically by showing that the application of equations (A-1) to the equivalent variational formulation of the two-region Sturm-Liouville Problem also results in equation (A-3). This will now be demonstrated.

To obtain the variational equivalent of equations (33) to (36), arbitrary variations of the functions E_{1n} and E_{2n} are considered. First, equation (33a) is multiplied by an arbitrary variation of E_{1n} , denoted by δE_{1n} .

Then, the following relation is used and multiplied by $x dx$:

$$(\delta E_{1n})(E_{1n}) = \frac{1}{2} \delta E_{1n}^2.$$

The result is integrated between $x = 0$ and $x = 1$, and the following is obtained:

$$\int_{x=0}^{x=1} \delta E_{1n} d(x E_{1n}) + \delta \left[\frac{\lambda_n^2}{2} \int_0^1 x g_1 E_{1n}^2 dx \right] = 0. \quad (B-1)$$

The first integral on the left is integrated by parts and manipulated as follows:

$$\begin{aligned} \int_{x=0}^{x=1} \delta E_{1n} d(x E_{1n}) &= E_{1n}(1) \delta E_{1n}(1) - \int_{x=0}^{x=1} x E_{1n} d(\delta E_{1n}) \\ &= E_{1n}(1) \delta E_{1n}(1) - \int_{x=0}^{x=1} x E_{1n} \delta(d E_{1n}) \\ &= E_{1n}(1) \delta E_{1n}(1) - \int_0^1 x E_{1n} \delta(E_{1n}) dx \\ &= E_{1n}(1) \delta E_{1n}(1) - \delta \left[\frac{1}{2} \int_0^1 x (E_{1n})^2 dx \right]. \end{aligned}$$

Thus, equation (B-1) can be written as

$$E'_{1n}(1)\delta E_{1n}(1) + \delta \left\{ \frac{1}{2} \int_0^1 \left[\lambda_n^2 g_1 E_{1n}^2 - (E'_{1n})^2 \right] x dx \right\} = 0. \quad (B-2)$$

In a similar manner, the variational equivalent of equation (33b),

$$E'_{2n}(1)\delta E_{2n}(1) + \delta \left\{ \frac{1}{2} \int_0^1 \left[\omega^2 \lambda_n^2 g_2 E_{2n}^2 - (E'_{2n})^2 \right] dx \right\} = 0, \quad (B-3)$$

is obtained. Now, from the coupling boundary conditions, equations (35) and (36),

$$\delta E_{2n}(1) = \delta E_{1n}(1) + K_w \delta E'_{1n}(1),$$

and

$$\begin{aligned} E'_{2n}(1)\delta E_{2n}(1) &= -K E'_{1n}(1) \left[\delta E_{1n}(1) + K_w \delta E'_{1n}(1) \right] \\ &= -K E'_{1n}(1) \delta E_{1n}(1) - \frac{1}{2} K K_w \delta \left[E'_{1n}(1) \right]^2. \end{aligned} \quad (B-4)$$

Combining equations (B-3) and (B-4) and using $\omega^2 = 1/2$ KH result in the following variational formulation of the two-region Sturm-Liouville Problem:

$$\delta \int_0^1 \left\{ \frac{\lambda_n^2}{2} \left[2xg_1 E_{1n}^2 + Hg_2 E_{2n}^2 \right] - \left[x(E'_{1n})^2 + \frac{1}{K} (E'_{2n})^2 + K_w (E'_{1n})^2 \right] \right\} dx = 0. \quad (B-5)$$

Equation (B-5) can be used as a basis for an approximation technique by choosing functional forms for the E_{1n} and E_{2n} involving arbitrary constants, as is done with the Rayleigh-Ritz method. "Best" values of the coefficients, as well as the λ_n , result from minimizing (or maximizing) the integral. For the application of interest here, equation (A-1) are chosen as the approximating functional forms.

The nomenclature is first simplified somewhat by dropping the subscript "n" and defining

$$I = \int_0^1 \left[2xg_1 E_1^2 + Hg_2 E_2^2 \right] dx,$$

and

$$J = \int_0^1 \left[x(E_1')^2 + \frac{1}{K} (E_2')^2 \right] dx.$$

Thus equation (B-5) can be written as

$$\delta \left\{ \frac{\lambda^2}{2} I - J - K_w [E_1'(1)]^2 \right\} = 0,$$

and with equations (A-1) used to represent E_1 and E_2 , the minimization requires that

$$\frac{\partial}{\partial a_k} \left\{ \frac{\lambda^2}{2} I - J - K_w [E_1'(1)]^2 \right\} = 0, \quad (B-6)$$

for $k = 0, 1, 2, \dots$

From equations (A-1), it is noted that

$$E_1^2 = \sum_{j,k} a_j a_k u_j u_k,$$

and

$$E_2^2 = \sum_{j,k} a_j a_k v_j v_k,$$

where the j,k notation under the summation sign means summation of all possible products with $j = 0, 1, 2, \dots$, and $k = 0, 1, 2, \dots$. Then, using P_{jk} as defined by equation (A-4) results in

$$I = \sum_{j,k} a_j a_k P_{jk}.$$

Similarly,

$$J = \sum_{j,k} a_j a_k \int_0^1 \left[x u_j' u_k' + \frac{1}{K} v_j' v_k' \right] dx.$$

But

$$\int_0^1 \left[x u_j' u_k' + \frac{1}{K} v_j' v_k' \right] dx = \int_{x=0}^{x=1} x u_j' du_k + \frac{1}{K} \int_{x=0}^{x=1} v_j' dv_k$$

(using $u'_k dx = du_k$, etc.),

$$= u'_j(1)u_k(1) + \frac{1}{K} v'_j(1)v_k(1) - \int_{x=0}^{x=1} u_k d(xu'_j) - \frac{1}{K} \int_{x=0}^{x=1} v_k dv'_j$$

(integration by parts, with $v'_k(0) = 0$)

$$= u'_j(1) [u_k(1) - v_k(1)] + \frac{\gamma_j^2}{2} \int_0^1 [2xu_k u_j + Hv_k v_j] dx$$

(using $Ku'_j(1) + v'_j(1) = 0$, $d(xu'_j) = -\gamma_j^2 u_j dx$, etc.)

$$= -K_w u'_j(1) u'_k(1) + \frac{\gamma_j^2}{2} \int_0^1 (2xu_k u_j + Hv_k v_j) dx$$

(using $K_w u'_k(1) + u_k(1) - v_k(1) = 0$)

$$= -K_w u'_j(1) u'_k(1), \quad j \neq k$$

(using the orthogonality conditions for the u_j and v_j)

$$= -K_w [u'_k(1)]^2 + \frac{\gamma_k^2}{2} N_k^{(0)}, \quad j = k,$$

where $N_k^{(0)}$ is given by equation (A-5). Thus,

$$\begin{aligned} J &= \sum_{k=0}^{\infty} a_k^2 \frac{\gamma_k^2}{2} N_k^{(0)} - K_w \sum_{j,k} a_j a_k u'_j(1) u'_k(1) \\ &= \sum_{k=0}^{\infty} a_k^2 \frac{\gamma_k^2}{2} N_k^{(0)} - K_w [E_1(1)]^2, \end{aligned}$$

and together with the expression derived for I , equation (B-6) can be written as

$$\frac{\partial}{\partial a_k} \left\{ \frac{\lambda^2}{2} \sum_{j,k} a_j a_k P_{jk} - \sum_{k=0}^{\infty} a_k^2 \frac{\gamma_k^2}{2} N_k^{(0)} \right\} = \lambda^2 \sum_{j=0}^{\infty} a_j P_{jk} - a_k \gamma_k^2 N_k^{(0)} = 0,$$

which is the same as equation (A-3).

APPENDIX C

Numerical Solutions of Equation (A-3)

The infinite set of simultaneous linear algebraic equations represented by equation (A-3) can be compactly expressed in matrix notation by defining the matrices

$$\mathbf{P} = [P_{kj}], \quad (\text{C-1})$$

$$\mathbf{A} = [\delta_{kj} N_k^{(0)} \gamma_k^2], \quad (\text{C-2})$$

$$\mathbf{X} = \{a_{nj}\}, \quad (\text{C-3})$$

where δ_{kj} is the Kronecker delta defined by

$$\delta_{kj} = 1, \quad k = j;$$

$$\delta_{kj} = 0, \quad k \neq j.$$

\mathbf{P} and \mathbf{A} are symmetric square matrices of infinite order; \mathbf{A} is positive, definite and diagonal; \mathbf{X} is an infinite column matrix or vector. The equivalent of equation (A-3) can now be written as

$$\lambda^2 \mathbf{P} \mathbf{X} = \mathbf{A} \mathbf{X}, \quad (\text{C-4})$$

with the characteristic equation

$$|\lambda^2 \mathbf{P} - \mathbf{A}| = 0$$

determining the eigenvalues λ_n^2 .

If equations (A-1) are truncated at $j = r$, the matrices of equation (C-4) are of finite order, $l + r$. Proven computer programs are available for the solution of equation (C-4) in this finite form, and thus this part of the numerical solution involves no special problems. Computations of the matrix elements, however, require numerical solutions of equation (59) for the γ_k , and numerical integrations of equation (A-4), and the question of the accuracy required for these quantities becomes important. Several test computations were made with r as large as 16. All computations reported in Tables I, II, and III used $r = 9$ or larger. It was found that the various quantities computed from the solution of equation (C-4) would, in most cases, converge quite rapidly up to r between 3 and 6, and then oscillate. A typical case is shown in Figure 7, where the computed value of $Nu_2(\infty)$ is plotted against r for $H = 1$, $K = 1$, and $K_w = 0$. For the cases explored, the computation of $Nu_2(\infty)$ was always the most sensitive, while computations of the λ_n^2 were the least sensitive. The important question is whether the oscillations are caused by small inaccuracies in the matrix elements, or are characteristic of the procedure when applied to "large perturbations."

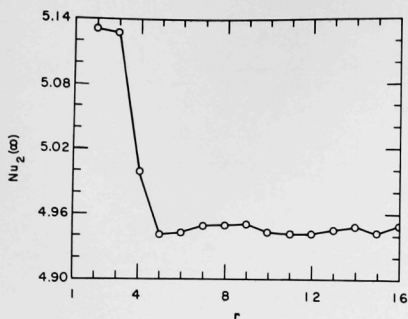


Fig. 7. Successive Approximations
for $Nu_2(\infty)$, ($H = 1$, $K = 1$, $K_W = 0$)

The matrix elements were computed to five significant digits after the decimal point, with uncertainty in the last one or two digits. Improvements in the accuracy of the matrix elements, while the same number of significant figures was maintained, did not remove the oscillations. Small differences in some of the computed quantities were noticed, however, suggesting that further improvements in accuracy might be helpful.

It was noted that the occurrence of oscillations could be roughly correlated with the occurrence of off-diagonal elements of \mathbf{P} becoming larger than diagonal elements in the same column. This suggests that the oscillations could be characteristic of the procedure when applied to "large perturbations" (i.e., when $1 - g_i$ is not sufficiently small) since $P_{kj} = 0$ and $P_{kk} = N_k^{(0)}$ when $g_i = 1$. It was also noted, as illustrated in Figure 7, that the oscillations appeared to be relatively uniform, whereas if small numerical inaccuracies were the cause, a more random behavior might be expected.

It was concluded that further study is required before the cause of the oscillations can be definitely determined, and that, whatever the cause, significant increases in computing time would be necessary to improve the accuracy of the results. Each case (H , K , K_W) with $r = 9$ required 10 minutes of IBM-704 machine time; for $r = 16$, nearly one half hour was necessary. Since results for at least 25 different cases are desired, the total machine time expenditure would be quite large. Thus, other methods of computation are being explored for future extensions of the problem.

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